

A CRITERION FOR DUALIZING MODULES

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ABSTRACT. We establish a characterization of dualizing modules among semidualizing modules. Let R be a finite dimensional commutative Noetherian ring with identity and C a semidualizing R -module. We show that C is a dualizing R -module if and only if $\mathrm{Tor}_i^R(E, E')$ is C -injective for all C -injective R -modules E and E' and all $i \geq 0$.

1. INTRODUCTION

Throughout this paper, R will denote a commutative Noetherian ring with non-zero identity. The injective envelope of an R -module M is denoted by $E_R(M)$.

A finitely generated R -module C is called *semidualizing* if the homothety map $R \rightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^i(C, C) = 0$ for all $i > 0$. Immediate examples of such modules are free R -modules of rank one. A semidualizing R -module C with finite injective dimension is called *dualizing*. Although R always possesses a semidualizing module, it does not possess a dualizing module in general. Keeping [BH, Theorem 3.3.6] in mind, it is straightforward to see that the ring R possesses a dualizing module if and only if it is Cohen-Macaulay and it is homomorphic image of a finite dimensional Gorenstein ring.

Let (R, \mathfrak{m}, k) be a local ring. There are several characterizations in the literature for a semidualizing R -module C to be dualizing. For instance, Christensen [C, Proposition 8.4] has shown that a semidualizing R -module C is dualizing if and only if the Gorenstein dimension of k with respect to C is finite. Also, Takahashi et al. [TTY, Theorem 1.3] proved that a semidualizing R -module C is dualizing if and only if every finitely generated R -module can be embedded in an R -module of finite C -dimension. Our aim in this paper is to give a new characterization for a semidualizing R -module C to be dualizing.

Let C be a semidualizing R -module. An R -module M is said to be *C -projective* (respectively *C -flat*) if it has the form $C \otimes_R U$ for some projective (respectively flat) R -module U . Also, a *C -free* R -module is defined as a direct sum of copies of C . We can see that every C -projective R -module is a direct summand of a C -free R -module and over a local ring every finitely generated C -flat R -module is C -free. Also, an R -module M is said to be *C -injective* if it has the form $\mathrm{Hom}_R(C, I)$ for some injective R -module I .

Yoneda raised a question of whether the tensor product of injective modules is injective. Ishikawa in [I, Theorem 2.4] showed that if $E_R(R)$ is flat, then $E \otimes_R E'$ is injective for all injective R -modules E and E' . Further, Enochs and Jenda [EJ, Theorem 4.1] proved that R is Gorenstein if and only if for every injective R -modules E and E' and any $i \geq 0$, $\mathrm{Tor}_i^R(E, E')$ is injective. We extend this result in terms of a semidualizing R -module. More precisely, for a semidualizing R -module C , we show that the following are equivalent (see Theorem 2.7):

- (i) $C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \mathrm{Spec} R$.

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(ii) For any prime ideal \mathfrak{p} of R and any $i \geq 0$,

$$\mathrm{Tor}_i^R(E_C(R/\mathfrak{p}), E_C(R/\mathfrak{p})) = \begin{cases} 0 & \text{if } i \neq \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}} \\ E_C(R/\mathfrak{p}) & \text{if } i = \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}}, \end{cases}$$

where $E_C(R/\mathfrak{p}) := \mathrm{Hom}_R(C, E_R(R/\mathfrak{p}))$.

(iii) For any C -injective R -modules E and E' and any $i \geq 0$, $\mathrm{Tor}_i^R(E, E')$ is C -injective.

2. THE RESULTS

Let \mathfrak{p} be a prime ideal of R . Recall that an R -module M is said to have property $t(\mathfrak{p})$ if for each $r \in R - \mathfrak{p}$, the map $M \xrightarrow{r} M$ is an isomorphism and if for each $x \in M$ we have $\mathfrak{p}^m x = 0$ for some $m \geq 1$. If an R -module M has $t(\mathfrak{p})$ -property, then it has the structure as an $R_{\mathfrak{p}}$ -module. It is known that $E_R(R/\mathfrak{p})$ has $t(\mathfrak{p})$ -property.

To prove Theorem 2.7, which is our main result, we shall need the following five preliminary lemmas.

Lemma 2.1. *Let C be a semidualizing R -module. Then the following statements hold true.*

- (i) $E_C(R/\mathfrak{p}) := \mathrm{Hom}_R(C, E_R(R/\mathfrak{p}))$ has $t(\mathfrak{p})$ -property for each $\mathfrak{p} \in \mathrm{Spec} R$.
- (ii) If \mathfrak{p} and \mathfrak{q} are two distinct prime ideals of R , then $\mathrm{Tor}_i^R(E_C(R/\mathfrak{p}), E_C(R/\mathfrak{q})) = 0$ for all $i \geq 0$.

Proof. (i) As $E_R(R/\mathfrak{p})$ has $t(\mathfrak{p})$ -property, one can easily check that for any finitely generated R -module M , the R -module $\mathrm{Hom}_R(M, E_R(R/\mathfrak{p}))$ has $t(\mathfrak{p})$ -property.

(ii) By (i) $E_C(R/\mathfrak{p})$ has $t(\mathfrak{p})$ -property and $E_C(R/\mathfrak{q})$ has $t(\mathfrak{q})$ -property. So, [EH, 5] implies that

$$\mathrm{Tor}_i^R(E_C(R/\mathfrak{p}), E_C(R/\mathfrak{q})) = 0$$

for all $i \geq 0$. □

Lemma 2.2. *Let (R, \mathfrak{m}, k) be a local ring, C a semidualizing R -module and I an Artinian C -injective R -module. Then $\mathrm{Hom}_R(I, E_R(k))$ is a finitely generated \widehat{C} -free \widehat{R} -module.*

Proof. Denote the functor $\mathrm{Hom}_R(-, E_R(k))$ by $(-)^{\vee}$. We have $I = \mathrm{Hom}_R(C, I')$ for some injective R -module I' . Clearly, $C \otimes_R I$ is also an Artinian R -module. Since

$$C \otimes_R I \cong C \otimes_R \mathrm{Hom}_R(C, I') \cong \mathrm{Hom}_R(\mathrm{Hom}_R(C, C), I') \cong I',$$

we deduce that I' is also Artinian. So, $I' \cong \bigoplus^n E_R(k)$ for some nonnegative integer n .

Now, one has

$$I^{\vee} = \mathrm{Hom}_R(C, I')^{\vee} \cong C \otimes_R I'^{\vee} \cong \bigoplus^n \widehat{C},$$

and so I^{\vee} is a finitely generated \widehat{C} -free \widehat{R} -module. □

In the next result, we collect some useful known properties of semidualizing modules. We may use them without any further comments.

Lemma 2.3. *Let C be a semidualizing R -module and $\underline{r} := r_1, \dots, r_n$ a sequence of elements of R . The following statements hold.*

- (i) $\mathrm{Supp}_R C = \mathrm{Spec} R$, and so $\dim_R C = \dim R$.
- (ii) If R is local, then \widehat{C} is a semidualizing \widehat{R} -module.
- (iii) \underline{r} is a regular R -sequence if and only if \underline{r} is a regular C -sequence.
- (iv) If \underline{r} is a regular R -sequence, then $C/(\underline{r})C$ is a semidualizing $R/(\underline{r})$ -module.
- (v) If R is local and \underline{r} is a regular R -sequence, then C is a dualizing R -module if and only if $C/(\underline{r})C$ is a dualizing $R/(\underline{r})$ -module.

Proof. (i) and (ii) follow easily by the definition of a semidualizing module.

(iii) and (iv) are hold by [S, Corollary 3.3.3].

(v) Assume that R is local and \underline{r} is a regular R -sequence. Then by (iv), $C/(\underline{r})C$ is a semidualizing $R/(\underline{r})$ -module. On the other hand, [BH, Corollary 3.1.15] yields that

$$\mathrm{id}_{\frac{R}{(\underline{r})}} \frac{C}{(\underline{r})C} = \mathrm{id}_R C - n.$$

This implies the conclusion. \square

In the proof of the following result, $R \ltimes C$ will denote the trivial extension of R by C . For any $R \ltimes C$ -module X , its Gorenstein injective dimension will be denoted by $\mathrm{Gid}_{R \ltimes C} X$. Also, we recall that for a module M over a local ring (R, \mathfrak{m}, k) , the width of M is defined by $\mathrm{width}_R M := \inf\{i \in \mathbb{N}_0 \mid \mathrm{Tor}_i^R(k, M) \neq 0\}$.

Lemma 2.4. *Let (R, \mathfrak{m}, k) be a local ring and C a semidualizing R -module. Then $E_C(k) \otimes_R E_C(k)$ is a non-zero C -injective R -module if and only if C is a dualizing R -module of dimension 0.*

Proof. Suppose that $E_C(k) \otimes_R E_C(k)$ is a non-zero C -injective R -module. As $E_C(k)$ is Artinian, by [KLS, Corollary 3.9] the length of $E_C(k) \otimes_R E_C(k)$ is finite. So, also, $(E_C(k) \otimes_R E_C(k))^\vee$ has finite length. Since

$$\mathrm{Hom}_R(E_C(k), \widehat{C}) \cong (E_C(k) \otimes_R E_C(k))^\vee,$$

by Lemma 2.2, we deduce that $\mathrm{Hom}_R(E_C(k), \widehat{C})$ is isomorphic to a direct sum of finitely many copies of \widehat{C} . This, in particular, implies that \widehat{C} has finite length. Thus Lemma 2.3 yields that

$$\dim R = \dim_R C = \dim_{\widehat{R}} \widehat{C} = 0,$$

and so, in particular, R is complete. Next, one has

$$\begin{aligned} \mathrm{Hom}_R(E_C(k), R) &\cong \mathrm{Hom}_R(E_C(k), \mathrm{Hom}_R(C, C)) \\ &\cong \mathrm{Hom}_R(C, \mathrm{Hom}_R(E_C(k), C)) \\ &\cong \bigoplus^n \mathrm{Hom}_R(C, C) \\ &\cong R^n \end{aligned}$$

for some $n > 0$. This, in particular, implies that

$$\mathrm{Ann}_R(\mathrm{Hom}_R(E_C(k), R)) = \mathrm{Ann}_R R.$$

Since R is Artinian, $\mathfrak{m}^t = 0$ and $\mathfrak{m}^{t-1} \neq 0$ for some $t > 0$. If for every $f \in \mathrm{Hom}_R(E_C(k), R)$, $\mathrm{im} f \subseteq \mathfrak{m}$, then $\mathfrak{m}^{t-1}f = 0$ so $\mathfrak{m}^{t-1} \mathrm{Hom}_R(E_C(k), R) = 0$ a contradiction. Thus there is an epimorphism $E_C(k) \rightarrow R \rightarrow 0$, and so R is a direct summand of $E_C(k)$. Next, [HJ1, Lemma 2.6] implies that R is a Gorenstein injective $R \ltimes C$ -module. This yields that C is a dualizing R -module, because by [HJ2, Proposition 4.5], one has

$$\mathrm{id}_R C \leq \mathrm{Gid}_{R \ltimes C} R + \mathrm{width}_R R.$$

Conversely, if C is a dualizing R -module of dimension 0, then $\dim R = 0$ by Lemma 2.3 (i). Hence, $E_R(k)$ is a dualizing R -module, and then by [BH, Theorem 3.3.4 (b)] we have $C \cong E_R(k)$. Thus

$$\begin{aligned} E_C(k) \otimes_R E_C(k) &\cong \mathrm{Hom}_R(E_R(k), E_R(k)) \otimes_R \mathrm{Hom}_R(E_R(k), E_R(k)) \\ &\cong R \otimes_R R \\ &\cong R \\ &\cong \mathrm{Hom}_R(C, E_R(k)), \end{aligned}$$

which is a non-zero C -injective R -module. \square

Remark 2.5. (See [B, (2.5)].) Let M be an R -module and let $r \in R$ be a non-unit which is a non-zero divisor of both R and M . Let $0 \rightarrow M \rightarrow I^0 \xrightarrow{d^0} I^1 \rightarrow \dots$ be a minimal injective resolution of M . Then there is a natural $R/(r)$ -isomorphism $M/(r)M \cong \text{Hom}_R(R/(r), \text{im } d^0)$ and

$$0 \rightarrow \text{Hom}_R(R/(r), I^1) \rightarrow \text{Hom}_R(R/(r), I^2) \rightarrow \dots$$

is a minimal injective resolution of the $R/(r)$ -module $M/(r)M$.

Next, we recall the definition of the notion of co-regular sequences. Let X be an R -module. An element r of R is said to be *co-regular* on X if the map $X \xrightarrow{r} X$ is surjective. A sequence r_1, \dots, r_n of elements of R is said to be a *co-regular sequence* on X if r_i is co-regular on $(0 :_M (r_1, \dots, r_{i-1}))$ for all $i = 1, \dots, n$.

The following result plays a crucial role in the proof of Theorem 2.7.

Lemma 2.6. *Let (R, \mathfrak{m}, k) be a local ring and C a semidualizing R -module. Let $r \in \mathfrak{m}$ be a non-zero divisor of R . Assume that r is co-regular on $\text{Tor}_i^R(E_C(k), E_C(k))$ for all i . Then for any $i \geq 0$, we have a natural \bar{R} -isomorphism*

$$\text{Tor}_{i-1}^{\bar{R}}(E_{\bar{C}}(k), E_{\bar{C}}(k)) \cong \text{Hom}_R(\bar{R}, \text{Tor}_i^R(E_C(k), E_C(k))),$$

where $\bar{R} := R/(r)$, $\bar{C} := C/(r)C$, $E_C(k) := \text{Hom}_R(C, E_R(k))$ and $E_{\bar{C}}(k) := \text{Hom}_{\bar{R}}(\bar{C}, E_{\bar{R}}(k))$.

Proof. Let $0 \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ be a minimal injective resolution of C . Then

$$\dots \rightarrow \text{Hom}_R(I^1, E_R(k)) \rightarrow \text{Hom}_R(I^0, E_R(k)) \rightarrow 0$$

is a flat resolution of $E_C(k)$. Applying $E_C(k) \otimes_R -$, we get the complex

$$\dots \rightarrow E_C(k) \otimes_R \text{Hom}_R(I^1, E_R(k)) \rightarrow E_C(k) \otimes_R \text{Hom}_R(I^0, E_R(k)) \rightarrow 0.$$

We will denote $E_C(k) \otimes_R \text{Hom}_R(I^i, E_R(k))$ by X_i and set

$$X_{\bullet} := \dots \rightarrow X_i \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow 0.$$

Then for each $i \geq 0$, we have $H_i(X_{\bullet}) = \text{Tor}_i^R(E_C(k), E_C(k))$.

By Remark 2.5,

$$0 \rightarrow \text{Hom}_R(\bar{R}, I^1) \rightarrow \text{Hom}_R(\bar{R}, I^2) \rightarrow \dots$$

is a minimal injective resolution of \bar{C} as an \bar{R} -module. So,

$$\dots \rightarrow \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^2), E_{\bar{R}}(k)) \rightarrow \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^1), E_{\bar{R}}(k)) \rightarrow 0$$

is a flat resolution of $E_{\bar{C}}(k)$ as an \bar{R} -module. Thus for each $i \geq 1$, the \bar{R} -module $\text{Tor}_{i-1}^{\bar{R}}(E_{\bar{C}}(k), E_{\bar{C}}(k))$ is isomorphic to the i th homology of the following complex

$$(\star) \dots \rightarrow E_{\bar{C}}(k) \otimes_{\bar{R}} \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^2), E_{\bar{R}}(k)) \rightarrow E_{\bar{C}}(k) \otimes_{\bar{R}} \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^1), E_{\bar{R}}(k)) \rightarrow 0.$$

We shall show that the later complex is isomorphic to the complex $Y_{\bullet} := \text{Hom}_R(\bar{R}, X_{\bullet})$.

Noting that $E_{\bar{R}}(k) \cong \text{Hom}_R(\bar{R}, E_R(k))$ and using Adjointness yields that

$$E_{\bar{C}}(k) = \text{Hom}_{\bar{R}}(\bar{C}, E_{\bar{R}}(k)) \cong \text{Hom}_R(\bar{R}, E_C(k)).$$

Hence for each $i \geq 0$, by using Adjointness, Hom-evaluation and Tensor-evaluation, one has the following natural \bar{R} -isomorphisms:

$$\begin{aligned}
E_{\bar{C}}(k) \otimes_{\bar{R}} \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^i), E_{\bar{R}}(k)) &\cong E_{\bar{C}}(k) \otimes_{\bar{R}} \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^i), \text{Hom}_R(\bar{R}, E_R(k))) \\
&\cong E_{\bar{C}}(k) \otimes_{\bar{R}} \text{Hom}_R(\text{Hom}_R(\bar{R}, I^i), E_R(k)) \\
&\cong E_{\bar{C}}(k) \otimes_{\bar{R}} (\bar{R} \otimes_R \text{Hom}_R(I^i, E_R(k))) \\
&\cong \text{Hom}_R(\bar{R}, E_C(k)) \otimes_R \text{Hom}_R(I^i, E_R(k)) \\
&\cong \text{Hom}_R(\bar{R}, E_C(k)) \otimes_R \text{Hom}_R(I^i, E_R(k)) \\
&\cong Y_i.
\end{aligned}$$

Note that $\text{Hom}_R(I^i, E_R(k))$ is a flat R -module. As r is a non-zero divisor of R , it is also a non-zero divisor of C . This implies that r is a non-zero divisor of I^0 , and so $\text{Hom}_R(\bar{R}, I^0) = 0$. Thus

$$Y_0 \cong E_{\bar{C}}(k) \otimes_{\bar{R}} \text{Hom}_{\bar{R}}(\text{Hom}_R(\bar{R}, I^0), E_{\bar{R}}(k)) = 0.$$

Therefore, the two complexes (\star) and Y_{\bullet} are isomorphic, and so we deduce that $\text{Tor}_{i-1}^{\bar{R}}(E_{\bar{C}}(k), E_{\bar{C}}(k)) = H_i(Y_{\bullet})$ for all $i \geq 0$.

Since r is a non-zero divisor of C , it is co-regular on $E_C(k)$, and so it is co-regular on X_i for all i . Thus, we can deduce the following exact sequence of complexes

$$0 \longrightarrow Y_{\bullet} \longrightarrow X_{\bullet} \xrightarrow{r} X_{\bullet} \longrightarrow 0.$$

It yields the following exact sequences of modules

$$\begin{aligned}
\cdots \longrightarrow \text{Tor}_{i+1}^R(E_C(k), E_C(k)) &\xrightarrow{r} \text{Tor}_{i+1}^R(E_C(k), E_C(k)) \longrightarrow \text{Tor}_{i-1}^{\bar{R}}(E_{\bar{C}}(k), E_{\bar{C}}(k)) \xrightarrow{f_i} \text{Tor}_i^R(E_C(k), E_C(k)) \\
&\xrightarrow{r} \text{Tor}_i^R(E_C(k), E_C(k)) \longrightarrow \cdots.
\end{aligned}$$

As r is a co-regular element on $\text{Tor}_i^R(E_C(k), E_C(k))$ for all i , we deduce that f_i is a monomorphism for all i . This implies our desired isomorphisms. \square

Theorem 2.7. *Let C be a semidualizing R -module. The following are equivalent:*

- (i) $C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Spec } R$.
- (ii) For any prime ideal \mathfrak{p} of R and any $i \geq 0$,

$$\text{Tor}_i^R(E_C(R/\mathfrak{p}), E_C(R/\mathfrak{p})) = \begin{cases} 0 & \text{if } i \neq \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}} \\ E_C(R/\mathfrak{p}) & \text{if } i = \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}}, \end{cases}$$

where $E_C(R/\mathfrak{p}) := \text{Hom}_R(C, E_R(R/\mathfrak{p}))$.

- (iii) For any C -injective R -modules E and E' and any $i \geq 0$, $\text{Tor}_i^R(E, E')$ is C -injective.

Proof. (i) \Rightarrow (ii) Let \mathfrak{p} be a prime ideal of R . There are natural $R_{\mathfrak{p}}$ -isomorphisms $E_C(R/\mathfrak{p}) \cong E_{C_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ and

$$\text{Tor}_i^R(E_C(R/\mathfrak{p}), E_C(R/\mathfrak{p})) \cong \text{Tor}_i^{R_{\mathfrak{p}}}(E_{C_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}), E_{C_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}))$$

for all $i \geq 0$. Hence, we can complete the proof of this part by showing that if C is a dualizing module of a local ring (R, \mathfrak{m}, k) , then

$$\text{Tor}_i^R(E_C(k), E_C(k)) = \begin{cases} 0 & i \neq \dim_R C \\ E_C(k) & i = \dim_R C. \end{cases}$$

Set $d := \dim_R C$. As C is a dualizing R -module, [BH, Theorem 3.3.10] implies that for any prime ideal \mathfrak{p} , one has

$$\mu^i(\mathfrak{p}, C) = \begin{cases} 0 & i \neq \text{ht } \mathfrak{p} \\ 1 & i = \text{ht } \mathfrak{p}. \end{cases}$$

So, if $I^\bullet = 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ is a minimal injective resolution of C , then $I^d \cong E_R(k)$ and for any $i \neq d$, $E_R(k)$ is not a direct summand of I^i . In particular, $\text{Hom}_R(R/\mathfrak{m}, I^i) = 0$ for all $i \neq d$. Now, $\text{Hom}_R(I^\bullet, E_R(k))$ is a flat resolution of $E_C(k)$. Clearly, one has

$$E_C(k) \otimes_R \text{Hom}_R(I^d, E_R(k)) \cong E_C(k) \otimes_R \widehat{R} \cong E_C(k).$$

Next, let $i \neq d$. Since $\text{Hom}_R(I^i, E_R(k))$ is a flat R -module, [M, Theorem 23.2 (ii)] implies that

$$\text{Ass}_R(E_C(k) \otimes_R \text{Hom}_R(I^i, E_R(k))) = \text{Ass}_R(R/\mathfrak{m} \otimes_R \text{Hom}_R(I^i, E_R(k))).$$

But,

$$R/\mathfrak{m} \otimes_R \text{Hom}_R(I^i, E_R(k)) \cong \text{Hom}_R(\text{Hom}_R(R/\mathfrak{m}, I^i), E_R(k)) = 0,$$

and so $E_C(k) \otimes_R \text{Hom}_R(I^i, E_R(k)) = 0$. Therefore, it follows that the complex $E_C(k) \otimes_R \text{Hom}_R(I^\bullet, E_R(k))$ has $E_C(k)$ in its d -place and 0 in its other places. Thus, we deduce that

$$\text{Tor}_i^R(E_C(k), E_C(k)) = H_i(E_C(k) \otimes_R \text{Hom}_R(I^\bullet, E(k))) = \begin{cases} 0 & i \neq d \\ E_C(k) & i = d. \end{cases}$$

(ii) \Rightarrow (iii) Let E be an injective R -module. Since $E \cong \bigoplus_{\mathfrak{p} \in \text{Spec } R} E_R(R/\mathfrak{p})^{\mu^0(\mathfrak{p}, E)}$ and C is finitely generated, we have

$$\text{Hom}_R(C, E) \cong \bigoplus_{\mathfrak{p} \in \text{Spec } R} E_C(R/\mathfrak{p})^{\mu^0(\mathfrak{p}, E)}.$$

As R is Noetherian, clearly any direct sum of C -injective R -modules is again C -injective, and so (ii) yields (iii) by Lemma 2.1 (ii).

(iii) \Rightarrow (i) It is easy to check that a given $R_{\mathfrak{p}}$ -module M is $C_{\mathfrak{p}}$ -injective if and only if it is the localization at \mathfrak{p} of a C -injective R -module. Thus, it is enough to show that if C is a semidualizing module of a local ring (R, \mathfrak{m}, k) such that $\text{Tor}_i^R(E, E')$ is C -injective for all C -injective R -modules E and E' and all $i \geq 0$, then C is dualizing.

Let $\underline{r} = r_1, \dots, r_d \in \mathfrak{m}$ be a maximal regular R -sequence. Then \underline{r} is also a regular C -sequence. It is easy to verify that \underline{r} is a co-regular sequence on any C -injective R -module, and consequently \underline{r} is a co-regular sequence on $\text{Tor}_i^R(E_C(k), E_C(k))$ for all $i \geq 0$. Letting $\bar{R} := R/(\underline{r})$ and $\bar{C} := C/(\underline{r})C$, by Lemma 2.3 (iv), it turns out that \bar{C} is a semidualizing \bar{R} -module. Making repeated use of Lemma 2.6, we can establish the following natural \bar{R} -isomorphism

$$E_{\bar{C}}(k) \otimes_{\bar{R}} E_{\bar{C}}(k) \cong \text{Hom}_{\bar{R}}(\bar{R}, \text{Tor}_d^R(E_C(k), E_C(k))).$$

So, $E_{\bar{C}}(k) \otimes_{\bar{R}} E_{\bar{C}}(k)$ is a \bar{C} -injective \bar{R} -module. Lemma 2.3 implies that

$$\text{depth}_{\widehat{\bar{R}}} \widehat{\bar{C}} = \text{depth}_{\bar{R}} \bar{C} = \text{depth}_{\bar{R}} \bar{R} = 0,$$

and so there are natural inclusion maps $k \xrightarrow{i} \bar{C}$ and $k \xrightarrow{j} \widehat{\bar{C}}$. By applying the functor $\text{Hom}_{\bar{R}}(-, E_{\bar{R}}(k))$ on i , we get an epimorphism $E_{\bar{C}}(k) \twoheadrightarrow k$. Next, by applying the functor $\text{Hom}_{\bar{R}}(-, \widehat{\bar{C}})$ on the later map, we see that

$$\text{Hom}_{\bar{R}}(E_{\bar{C}}(k) \otimes_{\bar{R}} E_{\bar{C}}(k), E_{\bar{R}}(k)) \cong \text{Hom}_{\bar{R}}(E_{\bar{C}}(k), \widehat{\bar{C}}) \neq 0.$$

Hence, $E_{\bar{C}}(k) \otimes_{\bar{R}} E_{\bar{C}}(k)$ is a non-zero \bar{C} -injective \bar{R} -module, and so Lemma 2.4 yields that \bar{C} is a dualizing \bar{R} -module. Now, by Lemma 2.3 (v), we deduce that C is a dualizing R -module. \square

We end the paper with the following immediate corollary.

Corollary 2.8. *Let R be a finite dimensional ring and C a semidualizing R -module. Then C is a dualizing R -module if and only if $\mathrm{Tor}_i^R(E, E')$ is C -injective for all C -injective R -modules E and E' and all $i \geq 0$.*

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